British Informatics Olympiad Final

28 – 30 March, 2008

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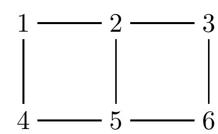
Colouring

A graph is a common data structure in computing which enables us to show the relationships between objects. A graph contains *nodes* (sometimes called vertices) each of which represents an object, and *edges* each of which connects two distinct nodes to indicate that they are related. Two nodes that are connected by an edge are said to be *adjacent*.

A common type of problem involves colouring the nodes so that each node is assigned a colour that is different to its adjacent nodes. In these questions, when we talk about colouring a graph, you must ensure that adjacent nodes *never* have the same colour.

Question 1

Suppose you can use the colours red, green and blue. In how many different ways can the following graph (the numbers represent nodes and the lines edges) be coloured, using only two of these colours, so that no adjacent nodes share the same colour? How about if all three colours are to be used?



A *path* is a sequence of nodes, with no node repeated, such that adjacent nodes in the path are connected by an edge; for example `1-2-5-6-3` is a path in the above graph. A *cycle* is a path, where the first node in the path and the last node in the path are also connected by an edge; for example `1-2-5-4` in the above graph.

Question 2

It is possible to colour the graph in question 1 using only two of the colours, but it is not always possible to colour a graph using only two colours. In general, what type of cycles cannot be coloured using only two colours?

Question 3

Suppose you are given a graph which can be coloured using only two colours. Outline an algorithm for colouring the graph using two colours.

Kempe Chaining

There is a famous theorem called the *Four-Colour Theorem* which says that any map can be coloured using at most four colours, so that no adjacent countries share the same colour. The equivalent theorem for graphs says that any *planar* graph, i.e. one which can be drawn so that no two edges cross-over, can also be coloured using at most four colours. This is very difficult to prove, but we will consider a simpler version where at most five colours can be used.

Question 4

Suppose that we have a planar graph where every node is adjacent to four or fewer other nodes. Outline an algorithm for colouring the graph using at most five colours.

It can be shown that every planar graph must have at least one node which is adjacent to five or fewer other nodes. Suppose we have a graph where one node (called N) is adjacent to five other nodes, and that the entire graph except for N has already been coloured with five colours, so that no two adjacent nodes share the same colour. If N is adjacent to four (or fewer) colours it is easy to colour, so let us assume that it is adjacent to nodes coloured with each of the five colours. Finally, suppose that we have drawn our planar graph so that no two edges cross-over and the nodes adjacent to N are, in clockwise order, coloured red, green, blue, yellow and purple. At the moment we are unable to colour N.

Let us call a path an *a-b path* if all the nodes on that path are coloured *a* or *b*.

Question 5

Suppose that no green-yellow path exists between the green and yellow nodes adjacent to N. If we took all the yellow and green nodes in the graph that are connected to the green node adjacent to N by a green-yellow path, we could swap their colours (yellow for green, and vice versa). How would this help us to colour N, and would the entire graph now have a valid colouring or would more work be required? Briefly justify your answer.

Question 6

Suppose that a green-yellow path did exist between the green and yellow nodes adjacent to N. Is it possible that there is a red-blue path between the red and blue nodes adjacent to N? Briefly justify your answer and explain how this might be used to modify the graph so that N can be coloured.

Question 7

Briefly outline how solutions to the above questions can be combined to make an algorithm for finding a way of colouring a planar graph with five colours.

The above algorithm was discovered in 1879 and, until 1890, it was thought that the same algorithm could be used to show that planar graphs could be coloured using only four colours. *Since the counter-example took several years to discover, you may wish to skip the next question.*

Question 8

Suppose we have a graph where N is adjacent to five other nodes, that the rest of the graph has been coloured with four colours, and that the colours of the adjacent nodes to N are (in clockwise order) red, green, blue, red and yellow. Briefly explain why the above algorithm (which reverses paths) might not work for this node.

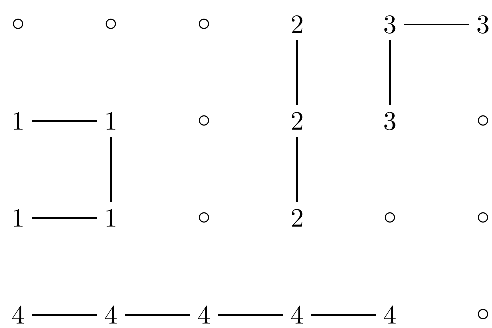
Circuit Testing

Modern electronic components are often fitted to circuit-boards and joined by conductive paths to form nets. Once a circuit-board has been produced it is desirable to test that there are no accidental conductive paths between components that should not be connected.

Question 9

One possible way to test a circuit-board is to run a separate test on all pairs of nets (which are not supposed to be connected). How many tests would be required if there are 100 nets on the circuit-board?

Let us consider a simplified type of circuit-board which consists of a rectangular grid of points. A net might consist of a group of connected adjacent components on the grid, where two components are adjacent if they are touching horizontally or vertically. The following example shows four different nets on the grid:



One method of circuit-board manufacture consists of producing the conductive paths and then placing the components onto the board. Let us suppose that the only type of manufacturing error consists of laying horizontal (or vertical) conductive paths where they should not exist; extra conductive paths which combine horizontal and vertical components will not occur. For example, the top-right part of net 1 might be incorrectly joined to the middle part of net 2 (an extra horizontal conductive path), but it would not be possible for the top-right part of net 1 to be incorrectly joined to the top (or bottom) part of net 2.

Question 10

In the above circuit-board it is possible that nets 1 and 3 are linked by an erroneous horizontal path, but it is not necessary to perform a test to see if these two nets have been connected. Why is this?

One way to reduce the number of tests would be to apply a *divide and conquer* algorithm as follows: the nets on the board could be split into two sets, each containing half the nets. Each set would then be tested to see if there are any errors; i.e. we would check if there were any undesirable conductive paths between any pair of nets in the first set, and then perform a similar check on the second set. We would then require an additional step to see whether there was an undesirable conductive path between a net in the first set and one in the second set.

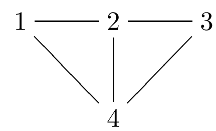
Question 11

When testing the sets we would apply our algorithm recursively. In other words, unless the set contained a single net (in which case no test is required), we would apply the same divide and conquer algorithm on that set. Suppose there are 2^n nets in total and that we can do the "additional step" (the comparison between two different sets) as a single test. How many tests would we apply, in total, using the divide and conquer algorithm?

Question 12

To perform the "additional step" as a single test it is necessary to perform some temporary re-wiring of the circuit-board; i.e. the temporary connecting together of different nets. Temporary re-wiring, since it does not involve printing conductive paths on the circuit-board, can connect any points on the circuit-board. What additional connections are required?

It is possible to use graph colouring algorithms to check a circuit-board, so that the number of tests required is independent of the number of nets. We can generate a graph from a circuit-board, where each node represents one of the nets, and two nets are connected by an edge if they can be reached by a single horizontal (or vertical) conductive path which does not intersect another net. For example, this is the graph that corresponds to the above circuit-board:



Question 13

Some circuit-boards produce graphs which are not planar. A graph (with 6 nodes) where each of nodes 1, 2 and 3 is connected to nodes 4, 5 and 6 is not planar. Draw a circuit-board which corresponds to such a graph.

Although not all circuit-boards produce graphs which are planar, there is a separate theorem which states that all circuit-board graphs can be coloured with at most 12 colours.

Question 14

Suppose that, prior to testing the circuit-board, we can do some temporary re-wiring. Explain how these temporary conductive paths, in conjunction with a way of colouring the circuit-board graph using at most 12 colours, can be used to reduce the number of tests to 66.

Question 15

Suppose that, in addition to adding temporary re-wiring before testing begins we are allowed to add additional re-wiring as testing progresses. Explain how the number of tests can be reduced to 11 *for any circuit-board*.

Question 16

Can the number of tests be reduced below 11 for all circuit-boards? Justify your answer.